Lecture 6

Time-Harmonic Fields, Complex Power

The analysis of Maxwell's equations can be greatly simplified by assuming the fields to be time harmonic, or sinusoidal (cosinusoidal). Electrical engineers use a method called phasor technique [33,53], to simplify equations involving time-harmonic signals. This is also a poor-man's Fourier transform [54]. That is one begets the benefits of Fourier transform technique without the knowledge of Fourier transform. Since only one time-harmonic frequency is involved, this is also called frequency domain analysis.[1] Phasors are represented in complex numbers. Therefore, the fields become complex in the frequency domain. From this, we will also discuss the concept of complex power.

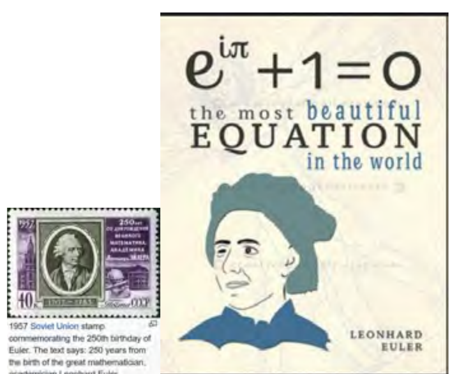

Figure 6.1: A commemorative stamp showing the contribution of Euler (courtesy of Wikipedia and Pinterest).

[1]It is simple only for linear systems: for nonlinear systems, such analysis can be quite unwieldy. But rest assured, as we will not discuss nonlinear systems in this course.

6.1 Time-Harmonic Fields—Linear Systems

To learn phasor technique, one makes use the formula due to Euler (1707–1783) (Wikipedia)[2]

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \tag{6.1.1}$$

where $j = \sqrt{-1}$ is an imaginary number.[3]

From Euler's formula, one gets

$$\cos\alpha = \Re e\left(e^{j\alpha}\right) \tag{6.1.2}$$

Hence, all time harmonic quantities can be written as

$$\begin{aligned} V(x,y,z,t) &= V'(x,y,z)\cos(\omega t + \alpha) &(6.1.3)\\ &= V'(\mathbf{r})\Re e(e^{j(\omega t+\alpha)}) &(6.1.4)\\ &= \Re e\left(V'(\mathbf{r})e^{j\alpha}e^{j\omega t}\right) &(6.1.5)\\ &= \Re e\left(\underset{\sim}{V}(\mathbf{r})e^{j\omega t}\right) &(6.1.6)\end{aligned}$$

Now $\underset{\sim}{V}(\mathbf{r}) = V'(\mathbf{r})e^{j\alpha}$ is a complex number called the phasor representation or phasor of $V(\mathbf{r},t)$, a time-harmonic quantity.[4] Here, the phase $\alpha = \alpha(\mathbf{r})$ can also be a function of position $\mathbf{r}$, or x, y, z. Consequently, any component of a field can be expressed as

$$E_x(x,y,z,t) = E_x(\mathbf{r},t) = \Re e\left[\underset{\sim}{E}_x(\mathbf{r})e^{j\omega t}\right] \tag{6.1.7}$$

The above can be repeated for y and z components. Compactly, for the x, y, and z components together, one can write

$$\mathbf{E}(\mathbf{r},t) = \Re e\left[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.8}$$

$$\mathbf{H}(\mathbf{r},t) = \Re e\left[\underset{\sim}{\mathbf{H}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.9}$$

where $\underset{\sim}{\mathbf{E}}$ and $\underset{\sim}{\mathbf{H}}$ are complex vector fields. Such phasor representations of time-harmonic fields simplify Maxwell's equations. For instance, if one writes

$$\mathbf{B}(\mathbf{r},t) = \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}\right) \tag{6.1.10}$$

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[2] As the stamp shows, Euler was blind in one eye.

[3] But lo and behold, in other disciplines, $\sqrt{-1}$ is denoted by "i", but "i" is too close to the symbol for current. So the preferred symbol for electrical engineering for an imaginary number is j: a quirkness of convention, just as positive charges do not carry current in a wire.

[4] We will use under tilde to denote a complex number or a phasor here, but this notation will be dropped later. Whether a variable is complex or real is clear from the context.

then

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) = \frac{\partial}{\partial t}\Re e\left[\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}\right]$$
$$= \Re e\left(\frac{\partial}{\partial t}\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}\right)$$
$$= \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right) \tag{6.1.11}$$

Therefore, a time derivative can be effected very simply for a time-harmonic field. One just needs to multiply $j\omega$ to the phasor representation of a field or a signal. Hence, given Faraday's law that

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} - \mathbf{M} \tag{6.1.12}$$

assuming that all quantities are time harmonic, then with (6.1.10) and what follows,

$$\mathbf{E}(\mathbf{r},t) = \Re e\left[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.13}$$

$$\mathbf{M}(\mathbf{r},t) = \Re e\left[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.14}$$

using (6.1.11) and the above into (6.1.12), one gets first

$$\nabla\times\mathbf{E}(\mathbf{r},t) = \Re e\left[\nabla\times\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.15}$$

and then

$$\Re e\left[\nabla\times\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}\right] = -\Re e\left[\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right] - \Re e\left[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}\right] \tag{6.1.16}$$

Since if

$$\Re e\left[A(\mathbf{r})e^{j\omega t}\right] = \Re e\left[B(\mathbf{r})e^{j\omega t}\right], \quad \forall t \tag{6.1.17}$$

then $A(\mathbf{r}) = B(\mathbf{r})$, it must be true from (6.1.16) that

$$\nabla\times\underset{\sim}{\mathbf{E}}(\mathbf{r}) = -j\omega\underset{\sim}{\mathbf{B}}(\mathbf{r}) - \underset{\sim}{\mathbf{M}}(\mathbf{r}) \tag{6.1.18}$$

Therefore, finding the phasor representation of an equation in the frequency domain is clear: whenever we have $\frac{\partial}{\partial t}$, we replace it by $j\omega$. Applying this methodically to the other Maxwell's equations, we have

$$\nabla\times\underset{\sim}{\mathbf{H}}(\mathbf{r}) = j\omega\underset{\sim}{\mathbf{D}}(\mathbf{r}) + \underset{\sim}{\mathbf{J}}(\mathbf{r}) \tag{6.1.19}$$

$$\nabla\cdot\underset{\sim}{\mathbf{D}}(\mathbf{r}) = \underset{\sim}{\varrho_e}(\mathbf{r}) \tag{6.1.20}$$

$$\nabla\cdot\underset{\sim}{\mathbf{B}}(\mathbf{r}) = \underset{\sim}{\varrho_m}(\mathbf{r}) \tag{6.1.21}$$

In the above, the phasors are functions of frequency. For instance, $\underset{\sim}{\mathbf{H}}(\mathbf{r})$ should rightly be written as $\underset{\sim}{\mathbf{H}}(\mathbf{r},\omega)$, but the ω dependence is implied.

6.2 Fourier Transform Technique

In the phasor representation, Maxwell's equations has no time derivatives; hence, the equations are simplified. We can also arrive at the above simplified equations using Fourier transform technique. To this end, we use Faraday's law as an example. By letting

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},\omega)e^{j\omega t}d\omega \tag{6.2.1}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{B}(\mathbf{r},\omega)e^{j\omega t}d\omega \tag{6.2.2}$$

$$\mathbf{M}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{M}(\mathbf{r},\omega)e^{j\omega t}d\omega \tag{6.2.3}$$

Substituting the above into Faraday's law given by (6.1.12), we get

$$\nabla\times\int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{E}(\mathbf{r},\omega) = -\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{B}(\mathbf{r},\omega) - \int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{M}(\mathbf{r},\omega) \tag{6.2.4}$$

Using the fact that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{B}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial t}e^{j\omega t}\mathbf{B}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t}j\omega\mathbf{B}(\mathbf{r},\omega) \tag{6.2.5}$$

and by exchanging the order of differentiation and integration, that

$$\nabla\times\int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t}\nabla\times\mathbf{E}(\mathbf{r},\omega) \tag{6.2.6}$$

Furthermore, using the fact that

$$\int_{-\infty}^{\infty} d\omega e^{j\omega t}A(\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t}B(\omega), \quad \forall t \tag{6.2.7}$$

implies that $A(\omega) = B(\omega)$, and using (6.2.5) and (6.2.6) in (6.2.4), and the property (6.2.7), one gets

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega) - \mathbf{M}(\mathbf{r},\omega) \tag{6.2.8}$$

These equations look exactly like the phasor equations we have derived previously, save that the field $\mathbf{E}(\mathbf{r},\omega)$, $\mathbf{B}(\mathbf{r},\omega)$, and $\mathbf{M}(\mathbf{r},\omega)$ are now the Fourier transforms of the field $\mathbf{E}(\mathbf{r},t)$,

$\mathbf{B}(\mathbf{r}, t)$, and $\mathbf{M}(\mathbf{r}, t)$. Moreover, the Fourier transform variables can be complex just like phasors. Repeating the exercise above for the other Maxwell's equations, we obtain equations that look similar to those for their phasor representations. Hence, Maxwell's equations can be simplified either by using phasor technique or Fourier transform technique. However, the dimensions of the phasors are different from the dimensions of the Fourier-transformed fields: $\underset{\sim}{\mathbf{E}}(\mathbf{r})$, a phasor, and $\mathbf{E}(\mathbf{r}, \omega)$, a Fourier transform, do not have the same dimension on closer examination.

6.3 Complex Power

Consider now that in the phasor representations, $\underset{\sim}{\mathbf{E}}(\mathbf{r})$ and $\underset{\sim}{\mathbf{H}}(\mathbf{r})$ are complex vectors, and their cross product, $\underset{\sim}{\mathbf{E}}(\mathbf{r}) \times \underset{\sim}{\mathbf{H}}^*(\mathbf{r})$, which still has the unit of power density, has a different physical meaning. First, consider the instantaneous Poynting's vector

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{6.3.1}$$

where all the quantities are real valued. Now, we can use phasor technique to analyze the above. Assuming time-harmonic fields, the above can be rewritten as

$$\begin{aligned}\mathbf{S}(\mathbf{r}, t) &= \Re e\left[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}\right] \times \Re e\left[\underset{\sim}{\mathbf{H}}(\mathbf{r})e^{j\omega t}\right] \\ &= \frac{1}{2}\left[\underset{\sim}{\mathbf{E}}e^{j\omega t} + (\underset{\sim}{\mathbf{E}}e^{j\omega t})^*\right] \times \frac{1}{2}\left[\underset{\sim}{\mathbf{H}}e^{j\omega t} + (\underset{\sim}{\mathbf{H}}e^{j\omega t})^*\right]\end{aligned} \tag{6.3.2}$$

where we have made use of the formula that

$$\Re e(Z) = \frac{1}{2}(Z + Z^*) \tag{6.3.3}$$

Then more elaborately, on expanding (6.3.2), we get

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}e^{2j\omega t} + \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^* + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}} + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}}^* e^{-2j\omega t} \tag{6.3.4}$$

Then rearranging terms and using (6.3.3) yield

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{2}\Re e\left[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*\right] + \frac{1}{2}\Re e\left[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}e^{2j\omega t}\right] \tag{6.3.5}$$

where the first term is independent of time, while the second term is sinusoidal in time. If we define a time-average quantity such that

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r}, t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{S}(\mathbf{r}, t)dt \tag{6.3.6}$$

then it is quite clear that the second term of (6.3.5) time-averages to zero since it is sinusoidal, and

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r}, t) \rangle = \frac{1}{2}\Re e\left[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*\right] \tag{6.3.7}$$

Therefore, in the phasor representation, the quantity

$$\underset{\sim}{\mathbf{S}} = \underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^* \tag{6.3.8}$$

is termed the complex Poynting's vector . The complex power density $\underset{\sim}{\mathbf{S}}$ (in watts per square meter), is energy density flow associated with it, and is associated with complex power.

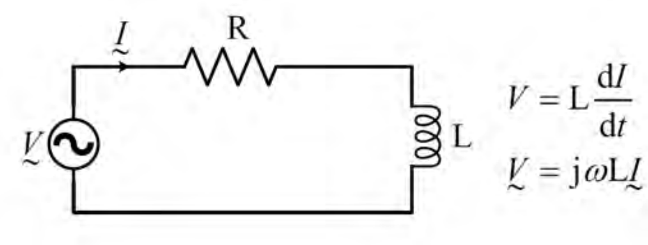

Figure 6.2: A simple circuit example to illustrate the concept of complex power in circuit theory. The voltage and current are out of phase which is a frequency-domain concept.

To understand what complex power is , it is fruitful if we revisit complex power [51, 55] in our circuit theory course. The circuit in Figure 6.2 can be easily solved by using phasor technique. The impedance of the circuit is $Z = R + j\omega L$. Hence,

$$\underset{\sim}{V} = (R + j\omega L)\underset{\sim}{I} \tag{6.3.9}$$

where $\underset{\sim}{V}$ and $\underset{\sim}{I}$ are the phasors of the voltage and current for time-harmonic signals. Just as in the electromagnetic case, the complex power in watts is taken to be

$$\underset{\sim}{P} = \underset{\sim}{V}\underset{\sim}{I}^* \tag{6.3.10}$$

But the instantaneous power is given by

$$P_{inst}(t) = V(t)I(t) \tag{6.3.11}$$

where $V(t) = \Re e\{\underset{\sim}{V}e^{j\omega t}\}$ and $I(t) = \Re e\{\underset{\sim}{I}e^{j\omega t}\}$. As shall be shown below,

$$P_{av} = \langle P_{inst}(t)\rangle = \frac{1}{2}\Re e\left[\underset{\sim}{P}\right] \tag{6.3.12}$$

It is clear that if $V(t)$ is sinusoidal, it can be written as

$$V(t) = V_0\cos(\omega t) = \Re e\left[\underset{\sim}{V}e^{j\omega t}\right] \tag{6.3.13}$$

where, without loss of generality, we assume that $\underset{\sim}{V} = V_0$. Then from (6.3.9), it is clear that $V(t)$ and $I(t)$ are not in phase. Namely that

$$I(t) = I_0\cos(\omega t + \alpha) = \Re e\left[\underset{\sim}{I}e^{j\omega t}\right] \tag{6.3.14}$$

where $\underset{\sim}{I} = I_0 e^{j\alpha}$. Then

$$\begin{aligned} P_{inst}(t) &= V_0 I_0 \cos(\omega t)\cos(\omega t + \alpha) \\ &= V_0 I_0 \cos(\omega t)\left[\cos(\omega t)\cos(\alpha) - \sin(\omega t)\sin\alpha\right] \\ &= V_0 I_0 \cos^2(\omega t)\cos\alpha - V_0 I_0 \cos(\omega t)\sin(\omega t)\sin\alpha \end{aligned} \tag{6.3.15}$$

It can be seen that the first term does not time-average to zero, but the second term, by letting $\cos(\omega t)\sin(\omega t) = 0.5\sin(2\omega t)$, does time-average to zero. Now taking the time average of (6.3.15), the time average of the first term involves the time average of $\cos^2(\omega t)$ which is 0.5, we get

$$P_{av} = \langle P_{inst} \rangle = \frac{1}{2} V_0 I_0 \cos\alpha = \frac{1}{2}\Re e\left[\underset{\sim}{V}\underset{\sim}{I}^*\right] \tag{6.3.16}$$

$$= \frac{1}{2}\Re e\left[\underset{\sim}{P}\right] \tag{6.3.17}$$

On the other hand, the reactive power

$$P_{reactive} = \frac{1}{2}\Im m\left[\underset{\sim}{P}\right] = \frac{1}{2}\Im m\left[\underset{\sim}{V}\underset{\sim}{I}^*\right] = \frac{1}{2}\Im m\left[V_0 I_0 e^{-j\alpha}\right] = -\frac{1}{2} V_0 I_0 \sin\alpha \tag{6.3.18}$$

One sees that amplitude of the time-varying term in (6.3.15) is precisely proportional to $\Im m\left[\underset{\sim}{P}\right]$.[5]

The reason for the existence of imaginary part of $\underset{\sim}{P}$ is because $V(t)$ and $I(t)$ are out of phase or $\underset{\sim}{V} = V_0$, but $\underset{\sim}{I} = I_0 e^{j\alpha}$. The reason for them being out of phase is because the circuit has a reactive part to it. Hence the imaginary part of complex power is also called the reactive power [36, 51, 55]. In a reactive circuit, the plots of the instantaneous power is shown in Figure 6.3. It is seen that when $\alpha \neq 0$, the instantaneous power can be negative. This means that the power is flowing from the load to the source instead of flowing from the source to the load at that instant. This happens only when the reactive power is nonzero or when a reactive component like an inductor or capacitor exists in the circuit. When a power company delivers power to our home, the power is complex because the current and voltage are not in phase. Even though the reactive power time-averages to zero, the power company still needs to deliver it to and from our home to run our washing machine, dish washer, fans, and air conditioner etc, and hence, charges us for it. Part of this power will be dissipated in the transmission lines that deliver power to our home. In other words, we have to pay for the use of imaginary power!

[5]Because that complex power is proportional to $\underset{\sim}{V}\underset{\sim}{I}^*$, it is the relative phase between $\underset{\sim}{V}$ and $\underset{\sim}{I}$ that matters. Therefore, α above is the relative phase between the phasor current and phasor voltage.

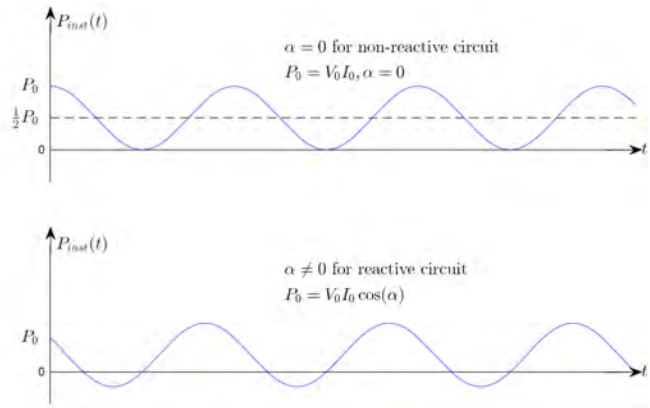

Figure 6.3: Plots of instantaneous power for when the voltage and the current is in phase ($\alpha = 0$), and when they are out of phase ($\alpha \neq 0$). In the out-of-phase case, there is an additional time-varying term that does not contribute to time-average power as shown in (6.3.15). Moreover, the instantaneous power can be negative.